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Generalized Vector Variational and Quasi-Variational Inequalities with Operator Solutions

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Abstract. In a recent paper, Domokos and Kolumbán introduced variational inequalities with operator solutions to provide a suitable unified approach to several kinds of variational inequality and vector variational inequality in Banach spaces. Inspired by their work, in this paper, we further develop the new scheme of vector variational inequalities with operator solutions from the single-valued case into the multi-valued one. We prove the existence of solutions of generalized vector variational inequalities with operator solutions and generalized quasi-vector variational inequalities with operator solutions. Some applications to generalized vector variational inequalities and generalized quasi-vector variational inequalities in a normed space are also provided.

Key words: vector variational inequality, C-pseudomonotone operator, generalized hemicontinuity, Fan–Browder fixed point theorem

1. Introduction

Since Giannessi [7] introduced the vector variational inequality, (shortly, VVI) in a finite dimensional Euclidean space, many authors have intensively studied (VVI) and its various extensions [1, 8, 9, 12, see also the references therein] in abstract spaces. Several authors have investigated relationships between (VVI) and vector optimization problems, vector complementarity problem [2, 11].

In a recent paper, Domokos and Kolumbán [5] gave an interesting interpretation of variational inequalities (VI) and (VVI) in Banach space settings in terms of variational inequalities with operator solutions (in short, OVVI). They first obtained an existence theorem of the solutions of (OVVI) using Fan's KKM Lemma [6], and then presented a general version of Yu and Yao [13, Theorem 3.3] in a Banach space as a main application and gave some other applications such as the solvability of variational inequality defined on Hausdorff topological vector space, and that of variational inequality on $L^{\infty}(\Omega)$. However, they dealt with only the single-valued operator.

Domokos and Kolumbán [5] designed (OVVI) to provide a suitable unified approach to several kinds of (VI) and (VVI) problems in Banach spaces, and successfully described those problems in a wider context of (OVVI). Inspired by their work, in this paper, we further develop the new scheme of (OVVI) from the single-valued case into the multi-valued one, and search some applications, from a theoretical point of view, to exploit the framework of (OVVI). To be more specific, we establish a multi-valued version of (OVVI) called the generalized vector variational inequality with operator solutions (in short, GOVVI). Also we introduce the quasi-version of (GOVVI) called the generalized vector quasi-variational inequality with operator solutions (in short, GOQVVI). As an application of (GOVVI), we provide a noncompact generalization of Konnov and Yao [9, Theorem 3.1] concerning a generalized (VVI) in a normed space (not necessarily Banach space). In addition, we deal with an existence theorem on (VVI) concerned with upper semicontinuity of multifunction instead of pseudo-monotonicity. This result is similar to Lai and Yao [10, Corollary 2.3]. As far as (GOQVVI) is concerned, we provide an existence of solution for a generalized vector QVI in a normed space (not necessarily Banach spaces). So we mainly focus on dealing with the existence of solutions of (GOVVI) and (GOQVVI) and their applications to (GVVI) and (GVQVI) in a normed space. In this respect, our work may be regarded as a first step toward the complete exploitation of the scheme of (OVVI) due to Domokos and Kolumbán [5].

As basic tools to obtain main results, we use a Fan–Browder type fixed point theorem due to Park [12, Theorem 5] and existence theorem of equilibrium for 1-person game due to Ding–Kim–Tan [4].

2. Preliminaries

Let E, F be Hausdorff topological vector spaces, and let X be a nonempty convex subset of E. A nonempty subset P of E is called *a convex cone* if

$$\lambda P \subset P \quad \text{for all} \quad \lambda > 0 \quad \text{and} \quad P + P = P.$$

Let $C_1 : X \rightrightarrows F$ be a multifunction such that for each $x \in X$, $C_1(x)$ is a convex cone in F with $\operatorname{int} C_1(x) \neq \emptyset$ and $C_1(x) \neq F$. Let $L(E, F)$ be the space of all continuous linear operators from E to F and $T_1 : X \rightrightarrows L(E, F)$ a multifunction.

Then T_1 is said to be

(1) C_1-*pseudomonotone* if for any $x, y \in X$ and for any $s \in T_1(x)$, we have

$$\langle s, y - x\rangle \notin -\operatorname{int} C_1(x) \text{ implies } \langle t, y - x\rangle \notin -\operatorname{int} C_1(x) \text{ for all } t \in T_1(y); \text{ and}$$

(2) *generalized hemicontinuous* if for any $x, y \in X$ and $\alpha \in [0, 1]$, the multifunction

$$\alpha \mapsto \langle T_1(x+\alpha(y-x)), y-x\rangle$$

is upper semicontinuous at 0^+, where

$$\langle T_1(x+\alpha(y-x)), y-x\rangle = \{\langle s, y-x\rangle \mid s \in T_1(x+\alpha(y-x))\}.$$

Now we pay our attention to generalized variational inequalities with operator solutions (in short, GOVVI). From now on, unless otherwise specified, we work under the following settings.

Let X' be a nonempty convex subset of $L(E, F)$ and $T: X' \rightrightarrows E$ be a multifunction. Let $C: X' \rightrightarrows F$ be a multifunction such that for each $f \in X'$, $C(f)$ is a convex cone in F with $0 \notin C(f)$. Then (GOVVI) is defined as follows:

$$\text{Find } f_0 \in X' \text{such that } \forall f \in X', \exists x \in T(f_0) \text{ with} \langle f - f_0, x\rangle \notin C(f_0). \tag{GOVVI}$$

When T is single-valued, (GOVVI) reduces to (OVVI) due to Domokos and Kolumbán [5]. As pointed out in [5], the notation (GOVVI) is motivated by the fact that the solutions are sought in the space of continuous linear operators. We also introduce the quasi-version of (GOVVI) called the generalized quasi-variational inequalities with operator solutions (in short, GOQVVI).

Let X' be a nonempty convex subset of $L(E, F)$ and $T: X' \rightrightarrows E$ be a multifunction. Let $C: X' \rightrightarrows F$ be a multifunction such that for each $f \in X'$, $C(f)$ is a convex cone in F with $0 \notin C(f)$, and let $A: X' \rightrightarrows X'$ be a multifunction. Then (GOQVVI) is defined as follows:

Find $f_0 \in X'$ such that $f_0 \in cl\, A(f_0)$ and

$$\forall f \in A(f_0), \quad \exists x \in T(f_0) \text{ with } \langle f - f_0, x\rangle \notin C(f_0). \tag{GOQVVI}$$

In regard to monotonicity and continuity of T, two analogous definitions to those of T_1 in the above are necessary; $T: X' \rightrightarrows E$ is said to be

(1)$'$ *C-pseudomonotone* if for any $f, g \in X'$ and for any $s \in T(f)$, we have

$$\langle g-f, s\rangle \notin C(f) \text{ implies } \langle g-f, t\rangle \notin C(f) \text{ for all } t \in T(g); \text{ and}$$

(2)$'$ *generalized hemicontinuous* if for any $f, g \in X'$ and $\alpha \in [0, 1]$, the multifunction

$$\alpha \mapsto \langle g-f, T(f+\alpha(g-f))\rangle$$

is upper semicontinuous at 0^+, where

$$\langle g-f, T(f+\alpha(g-f))\rangle = \{\langle g-f, s\rangle \mid s \in T(f+\alpha(g-f))\}.$$

In order to prove our main result, we need the following fixed point theorem which is a particular form of Park [12, Theorem 5].

LEMMA 2.1. *Let X be a nonempty convex subset of a real (not necessarily) Hausdorff topological vector space E, K a nonempty compact subset of X. Let A, $B: X \rightrightarrows X$ be two multifunctions. Suppose that*

(i) *for each $x \in X$, $Ax \subset Bx$;*
(ii) *for each $x \in X$, Bx is convex;*
(iii) *for each $x \in K$, Ax is nonempty;*
(iv) *for each $y \in X$, $A^{-1}y = \{x \in X \mid y \in Ax\}$ is open in X; and*
(v) *for each finite subset N of X, there exists a nonempty compact convex subset L_N of X containing N such that for each $x \in L_N \backslash K$, $Ax \cap L_N \neq \emptyset$.*

Then B has a fixed point x_0; that is, $x_0 \in Bx_0$.

We also need the following lemma, which is a special case of Theorem 2 of Ding-Kim-Tan [4].

LEMMA 2.2. *Let $\Gamma = (X, A, P)$ be an 1-person game such that*

(1) *X is a nonempty compact convex subset of a Hausdorff topological vector space,*
(2) *$A: X \rightrightarrows X$ is a multifunction such that for each $x \in X$, $A(x)$ is nonempty convex and for each $y \in X$, $A^{-1}(y)$ is open in X,*
(3) *the multifunction $cl\, A: X \rightrightarrows X$ is upper semicontinuous,*
(4) *the multifunction $P: X \rightrightarrows X$ is such that $P^{-1}(y)$ is open in X for each $y \in X$,*
(5) *for each $x \in X$, $x \notin co\, P(x)$, where $co\, P(x)$ denotes the convex hull of $P(x)$. Then Γ has an equilibrium choice $\hat{x} \in X$; i.e.,*

$$\hat{x} \in cl\, A(\hat{x}) \quad and \quad A(\hat{x}) \cap P(\hat{x}) = \emptyset.$$

3. Generalized Vector Variational Inequality with Operator Soultions

We begin with the following lemma to get the main result.

LEMMA 3.1. *Let $T: X' \rightrightarrows E$ be a C-pseudomonotone and generalized hemicontinuous multifunction with $T(f) \neq \emptyset$ for all $f \in X'$. Let $W: X' \rightrightarrows F$ be* *defined by* $W(f)=F\backslash C(f)$ *such that the graph* $Gr(W)$ *of* W *is closed in* $X'\times F$, *where* $L(E,F)$ *is endowed with the topology of pointwise convergence. Then the following two problems are equivalent:*

(i) *Find* $f\in X'$ *such that* $\forall g\in X',\ \exists x\in T(f)$ *with* $\langle g-f,x\rangle\notin C(f)$.
(ii) *Find* $f\in X'$ *such that* $\forall g\in X',\ \forall x\in T(g),\ \langle g-f,x\rangle\notin C(f)$.

Proof. (i) $\Rightarrow$ (ii). This is immediate from the C-pseudomonotonicity of T.
(ii) $\Rightarrow$ (i). Let $f\in X'$ be a solution of (ii). Suppose by contradiction that f is not a solution of (i). Then there exists $g_0\in X'$ such that

$$\forall x\in T(f),\quad \langle g_0-f,x\rangle\in C(f). \tag{3.1}$$

Since f is a solution of (ii), we have, for each $t\in(0,1)$,

$$\langle tg_0+(1-t)f-f,x_t\rangle\notin C(f)\quad\text{for all}\quad x_t\in T(f+t(g_0-f)).$$

Hence

$$\langle g_0-f,x_t\rangle\notin C(f)\quad\text{for all}\quad x_t\in T(f+t(g_0-f)). \tag{3.2}$$

As T is generalized hemicontinuous, the multifunction $H:[0,1]\rightrightarrows F$ defined by $H(t)=\langle g_0-f,T(f+t(g_0-f))\rangle$ is upper semicontinuous at 0^+. It follows from (3.1) that

$$H(0)=\langle g_0-f,T(f)\rangle\subset C(f).$$

Observe that the closedness of $Gr(W)$ implies that of $W(f)$ for every $f\in X'$. Thus $C(f)$ is open in F for every $f\in X'$. Hence there exists $\bar t\in(0,1)$ such that

$$H(t)=\langle g_0-f,T(f+t(g_0-f))\rangle\subset C(f)\quad\text{for all}\quad t\in(0,\bar t),$$

which contradicts (3.2). This completes the proof. □

Using Lemma 3.1, we first prove the following which is a multi-valued version of (OVVI) in [5].

THEOREM 3.1. *Let* $T:X'\rightrightarrows E$ *be a* C*-pseudomonotone and generalized hemicontinuous multifunction with* $T(f)\neq\emptyset$ *for all* $f\in X'$. *Let* $W:X'\rightrightarrows F$ *be defined by* $W(f)=F\backslash C(f)$ *such that the graph* $Gr(W)$ *of* W *is closed in* $X'\times F$, *where* $L(E,F)$ *is endowed with the topology of pointwise convergence. Let* K' *be a nonempty compact subset of* X'. *Assume that for each* *finite subset N' of X', there exists a nonempty compact convex subset $L_{N'}$* of *X' containing N' such that for each $f \in L_{N'} \setminus K'$, there exists $g \in L_{N'}$ satisfying*

$$\langle g - f, x \rangle \in C(f) \quad \textit{for some} \quad x \in T(g).$$

Then (GOVVI) *is solvable.*

Proof. First note that $L(E, F)$ equipped with the topology of pointwise convergence is a Hausdorff t.v.s. We define two multifunctions $A, B\colon X' \rightrightarrows X'$ to be

$$A(f) := \{g \in X' \mid \exists x \in T(g), \text{ such that } \langle g - f, x \rangle \in C(f)\},$$
$$B(f) := \{g \in X' \mid \forall x \in T(f), \quad \langle g - f, x \rangle \in C(f)\}.$$

The proof is organized in the following parts.

(i) Since T is C-pseudomonotone, we have $A(f) \subset B(f)$ for all $f \in X'$.

(ii) For each $f \in X'$, $B(f)$ is convex. Indeed, let g_1 and g_2 be in $B(f)$. For all $t \in [0, 1]$ and $x \in T(f)$, we have

$$\langle tg_1 + (1-t)g_2 - f, x \rangle = t\langle g_1 - f, x \rangle + (1-t)\langle g_2 - f, x \rangle \in C(f),$$

which implies that $tg_1 + (1-t)g_2 \in B(f)$. Hence $B(f)$ is convex.

(iii) Clearly B has no fixed point because $0 \notin C(f)$ for all $f \in X'$.

(iv) For each $g \in X'$, $A^{-1}(g)$ is open in X'. In fact, let $\{f_\lambda\}$ be a net in $(A^{-1}(g))^c$ convergent to $f \in X'$. Then $g \notin A(f_\lambda)$ and hence for each $x \in T(g)$,

$$\langle g - f_\lambda, x \rangle \notin C(f_\lambda).$$

Thus $\langle g - f_\lambda, x \rangle \in W(f_\lambda)$. Since $(f_\lambda, \langle g - f_\lambda, x \rangle) \in \mathrm{Gr}(W)$ and $L(E, F)$ is endowed with the topology of pointwise convergence, by virtue of the closedness of $\mathrm{Gr}(W)$, we have $(f, \langle g - f, x \rangle) \in \mathrm{Gr}(W)$, that is, $\langle g - f, x \rangle \notin C(f)$ for every $x \in T(g)$. Hence $g \notin A(f)$, so $f \in (A^{-1}(g))^c$. This shows that $(A^{-1}(g))^c$ is closed, therefore $A^{-1}(g)$ is open in X'.

(v) By the given hypothesis, we know that for each finite subset N' of X', there exists a nonempty compact convex subset $L_{N'}$ of X' containing N' such that for each $f \in L_{N'} \setminus K'$, there exists $g \in L_{N'}$ satisfying $g \in A(f)$, hence $L_{N'} \cap A(f) \neq \emptyset$.

(vi) From (i)–(v), we see, by Lemma 2.1, there must be an $f_0 \in K'$ such that $A(f_0) = \emptyset$, namely,

$$\langle g - f_0, x \rangle \notin C(f_0) \quad \text{for any} \quad g \in X', \; x \in T(g).$$

It follows from Lemma 3.1 that f_0 is a solution of (GOVVI). This completes the proof. □

To obtain an application of Theorem 3.1, the following lemma is necessary. For reader's convenience, we provide a detailed proof.

LEMMA 3.2. *Let E and F be normed spaces. Then $E^* = \{g \circ L \mid g \in F^*$ and $L \in L(E, F)\}$, where E^* and F^* denote the continuous dual spaces of E and F, respectively.*

Proof. It suffices to show that every continuous linear functional $h \in E^*$ can be represented by

$$h(x) = g \circ L(x) \quad \text{for all } x \in E,$$

where $g \in F^*$ and $L \in L(E, F)$. Let $\text{Ker}h = M$. Choose $x_0 \in E$ with $h(x_0) = 1$ and $y_0 \in F$ with $\|y_0\| = 1$. Then

$$E = \bigcup_{\alpha \in R} \{\alpha x_0 + M\}.$$

Put $Q = \{\alpha y_0 \mid \alpha \in R\}$ the one-dimensional subspace of F generated by y_0. We define a mapping $L: E \to Q$ by

$$L(\alpha x_0 + m) = \alpha y_0, \quad \text{where } m \in M.$$

Clearly L is linear. Also define an isomorphism $j: Q \to R$ to be $j(\alpha y_0) = \alpha$. Then it is easy to check that $h = j \circ L$. This implies that $L = j^{-1} \circ h: E \to Q$ is continuous. In addition, we may assume that $L \in L(E, F)$. By the Hahn–Banach Theorem [3, Corollary 6.5, p. 81], there exists a continuous linear functional $g \in F^*$, such that $g(y) = j(y)$ for all $y \in Q$ and $\|g\| = \|j\|$. Therefore we have $h = g \circ L$, as desired. This completes the proof. □

As an application of Theorem 3.1 in multi-valued settings, we shall prove the existence of a solution of a generalized (VVI) in a normed space.

THEOREM 3.2. *Let Y and Z be two normed spaces. Let X be a nonempty convex subset of Y and $C_1: X \rightrightarrows Z$ be a multifunction such that for each $x \in X$, $C_1(x)$ is a convex cone in Z with $\text{int}\, C_1(x) \neq \emptyset$ and $C_1(x) \neq Z$. Let $T_1: X \rightrightarrows L(Y, Z)$ be a C_1-pseudomonotone and generalized hemicontinuous multifunction with nonempty values. Let $W_1: X \rightrightarrows Z$ be defined by $W_1(x) = Z \setminus -\text{int}\, C_1(x)$ such that the graph $Gr(W_1)$ of W_1 is weakly closed in $X \times Z$. Assume that K is a nonempty weakly compact subset of X and for each finite subset N of X, there exists a nonempty weakly compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there exists $y \in L_N$ satisfying*

$$\langle s, y-x\rangle \in -\operatorname{int} C_1(x) \quad \textit{for some } s \in T_1(y).$$

Then there exists $x_0 \in X$ *such that*

$$\forall x \in X, \quad \exists t \in T_1(x_0) \quad \textit{with } \langle t, x-x_0\rangle \notin -\operatorname{int} C_1(x_0).$$

Proof. We consider $E = L(Y, Z)$ as the normed space of the continuous linear operators between Y and Z with the usual norm, and $F = (Z,\ weak)$ as the Hausdorff locally convex topological vector space Z endowed with the weak topology. Define a mapping $\phi: Y \to L(E, F)$ by $\phi(x) = f_x$ where $f_x(l) = \langle l, x\rangle$ for all $l \in E$. This ϕ is linear and 1–1. Moreover, ϕ is a homeomorphism from $(X, weak)$ onto $\phi(X) = X'$ with the subspace topology of $L(E, F)$ equipped with the topology of pointwise convergence. In fact, let $\{x_i\}$ be a net in X weakly convergent to $x \in X$. Then for each $l \in E = L(Y, Z)$, $\langle l, x_i\rangle \rightharpoonup \langle l, x\rangle$ because $l: (Y,\ weak) \to (Z,\ weak)$ is continuous (see Conway [3, Theorem 1.1, p. 171]). This implies that $\forall l \in E,\ f_{x_i}(l) \to f_x(l)$ in F. Thus $f_{x_i} \to f_x$ in $L(E, F)$, which means that ϕ is continuous. Conversely, let $\{f_{x_i}\}$ be a net in X' convergent to $f_x \in X'$. Then for each $l \in E,\ f_{x_i}(l) \to f_x(l)$ in F, hence $\langle l, x_i\rangle \rightharpoonup \langle l, x\rangle$. So $\langle w^* \circ l, x_i\rangle \to \langle w^* \circ l, x\rangle$ in R for all $l \in L(Y, Z)$ and $w^* \in Z^*$ the dual space of Z. Thus, by Lemma 3.2, we have

$$\langle y^*, x_i\rangle \to \langle y^*, x\rangle \ \forall y^* \in Y^*.$$

This shows that $x_i \rightharpoonup x$ in X. Therefore ϕ^{-1} is continuous.

Now we define $T: X' \rightrightarrows E$, $C: X' \rightrightarrows F$ and $W: X' \rightrightarrows F$ as follows:

$$T(f_x) := T_1(x),$$
$$C(f_x) := -\operatorname{int} C_1(x),$$
$$W(f_x) := W_1(x),$$

where $\operatorname{int} C_1(x)$ is the interior of $C_1(x)$ in the normed space Z. Then $0 \notin C(f_x)$ because $\operatorname{int} C_1(x)$ is a proper convex cone of Z. The proof is organized in the following parts.

(i) If D is a weakly compact subset of X, then $\phi(D)$ is compact in $L(E, F)$ because ϕ is a homeomorphism.

(ii) C_1-pseudomonotonicity of T_1 implies the C-pseudomonotonicity of T. In fact, for any $f_x,\ f_y \in X'$ and $s \in T(f_x) = T_1(x)$,

$$\begin{aligned}\langle f_y - f_x, s\rangle \notin C(f_x) &\Rightarrow \langle s, y-x\rangle \notin -\operatorname{int} C_1(x)\\ &\Rightarrow \langle t, y-x\rangle \notin -\operatorname{int} C_1(x) \quad \text{for all} \quad t \in T_1(y) = T(f_y)\\ &\Rightarrow \langle f_y - f_x, t\rangle \notin C(f_x) \quad \text{for all} \quad t \in T(f_y).\end{aligned}$$

(iii) The generalized hemicontinuity of T_1 amounts to that of T. Actually, for any $f_x,\ f_y \in X'$ and $\alpha \in [0, 1]$,

$$\alpha \mapsto \langle f_y - f_x, T(f_x + \alpha(f_y - f_x))\rangle = \langle T_1(x + \alpha(y - x)), y - x\rangle$$

is upper semicontinuous at 0^+.

(iv) W has a closed graph in $X' \times F$, where $L(E, F)$ is endowed with the topology of pointwise convergence. Indeed, let $\{f_{x_i}\}$ be a net in X' convergent to $f_x \in X'$. Clearly, $x_i \rightharpoonup x$ in X because ϕ is a homeomorphism. Let $w_i \in W(f_{x_i})$ such that $w_i \to w$ in F. The weak closedness of the graph $Gr(W_1)$ of W_1 in $X \times Z$ and the equivalence $W(f_x) = W_1(x)$ yield that $(x, w) \in Gr(W_1)$, i.e., $w \in W_1(x)$, hence $w \in W(f_x)$.

(v) Put $K' = \phi(K)$. By the hypothesis and (i), it can be readily checked that for each finite subset N' of X', there exists a nonempty compact convex subset $L_{N'}$ of X' containing N' such that for each $f_x \in L_{N'} \backslash K'$, there exists $f_y \in L_{N'}$ satisfying

$$\langle f_y - f_x, s\rangle \in C(f_x) \quad \text{for some} \quad s \in T(f_y).$$

It follows directly from Theorem 3.1 that there exists $f_{x_0} \in X'$ such that for each $f_x \in X'$, there is $t \in T(f_{x_0})$ with $\langle f_x - f_{x_0}, t\rangle \notin C(f_{x_0})$. Therefore, there exists $x_0 \in X$ such that

$$\forall x \in X, \quad \exists t \in T_1(x_0) \quad \text{with } \langle t, x - x_0\rangle \notin -\operatorname{int} C_1(x_0).$$

This completes the proof. □

REMARK 3.1. Theorem 3.2 is a noncompact generalization of Konnov and Yao [9, Theorem 3.1] in normed spaces (not necessarily Banach spaces) without assuming the convex cone $C_1(x)$ being closed.

Now we are interested in (GOVVI) concerned with the upper semicontinuity of T instead of pseudomonotonicity and hemicontinuity. To this end, we replace the topology of pointwise convergence by that of bounded convergence on $L(E, F)$.

THEOREM 3.3. *Suppose that $L(E, F)$ is endowed with the topology of bounded convergence. Let $T: X' \rightrightarrows E$ be an upper semicontinuous multifunction such that $T(f)$ is a nonempty compact subset of E for all $f \in X'$, and the range $T(X')$ is contained in a compact subset of E. Let $W: X' \rightrightarrows F$ be defined by $W(f) = F \setminus C(f)$ such that the graph $Gr(W)$ of W is closed in $X' \times F$. Let K' be a nonempty compact subset of X'. Assume that for each* *finite subset N' of X', there exists a nonempty compact convex subset $L_{N'}$ of X' containing N' such that for each $f \in L_{N'} \backslash K'$, there exists $g \in L_{N'}$ satisfying*

$$\langle g - f, x \rangle \in C(f) \quad for\ all\ x \in T(f).$$

Then (GOVVI) is solvable.

Proof. We define a multifunction $B: X' \rightrightarrows X'$ to be

$$B(f) := \{g \in X' \mid \forall x \in T(f), \quad \langle g - f, x \rangle \in C(f)\}.$$

(i) As seen in the proof of Theorem 3.1, $B(f)$ is convex for all $f \in X'$ and B has no fixed point.

(ii) For each $g \in X'$, $B^{-1}(g)$ is open in X'. In fact, let $\{f_\lambda\}$ be a net in $(B^{-1}(g))^c$ convergent to $f \in X'$. Then $g \notin B(f_\lambda)$ so that there exists an $x_\lambda \in T(f_\lambda)$ satisfying

$$\langle g - f_\lambda, x_\lambda \rangle \notin C(f_\lambda).$$

Thus $\langle g - f_\lambda, x_\lambda \rangle \in W(f_\lambda)$. Since $T(X')$ is contained in a compact subset, we may assume without loss of generality that $x_\lambda \to x$ for some $x \in E$. Observe that $\mathrm{Gr}(T)$ is closed, so $x \in T(f)$ because T is upper semicontinuous and compact-valued. Since $(f_\lambda, \langle g - f_\lambda, x_\lambda \rangle) \in \mathrm{Gr}(W)$ and $L(E, F)$ is endowed with the topology of bounded convergence, by virtue of the closedness of $\mathrm{Gr}(W)$, we have $(f, \langle g - f, x \rangle) \in \mathrm{Gr}(W)$, that is, $\langle g - f, x \rangle \notin C(f)$. Hence $g \notin B(f)$, i.e., $f \in (B^{-1}(g))^c$. This shows that $(B^{-1}(g))^c$ is closed, therefore $B^{-1}(g)$ is open in X'.

(iii) By the hypothesis, we know that for each finite subset N' of X', there exists a nonempty compact convex subset $L_{N'}$ of X' containing N' such that for each $f \in L_{N'} \setminus K'$, there exists $g \in L_{N'}$ satisfying $g \in B(f)$, hence $L_{N'} \cap B(f) \neq \emptyset$.

(iv) From (i) to (iii), we see, by Lemma 2.1 in case of $A = B$, there must be an $f_0 \in K'$ such that $B(f_0) = \emptyset$. This means that for any $g \in X'$, there exists an $x \in T(f_0)$ such that

$$\langle g - f_0, x \rangle \notin C(f_0).$$

Therefore f_0 is a solution of (GOVVI). This completes the proof. □

As a direct consequence of Theorem 3.3, we obtain the following.

THEOREM 3.4. *Let Y and Z be two normed spaces. Let X be a nonempty convex subset of Y and $C_1: X \rightrightarrows Z$ be a multifunction such that for each $x \in X$, $C_1(x)$ is a convex cone in Z with $\mathrm{int}\, C_1(x) \neq \emptyset$ and $C_1(x) \neq Z$.*

Let $T_1: X \rightrightarrows L(Y, Z)$ be an upper semicontinuous multifunction with nonempty compact values and the range $T_1(X)$ be contained in a compact subset of $L(Y, Z)$, where $L(Y, Z)$ is the normed space of the continuous linear operators between Y and Z with the usual norm. Let $W_1: X \rightrightarrows Z$ be defined by $W_1(x) = Z \backslash -\text{int}\, C_1(x)$ such that the graph $Gr(W_1)$ of W_1 *is closed in $X \times Z$. Assume that K is a nonempty compact subset of X and for each finite subset N of X, there exists a nonempty compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there exists $y \in L_N$ satisfying*

$$\langle s, y - x \rangle \in -\text{int}\, C_1(x) \quad \textit{for all} \quad s \in T_1(x).$$

Then there exists $x_0 \in X$ such that

$$\forall x \in X, \quad \exists t \in T_1(x_0) \quad \textit{with} \ \langle t, x - x_0 \rangle \notin -\text{int}\, C_1(x_0).$$

Proof. We consider $E = L(Y, Z)$ as the normed space of the continuous linear operators between Y and Z with the usual norm, and $F = (Z, \|\cdot\|)$. Define a mapping $\phi: Y \to L(E, F)$ by $\phi(x) = f_x$, where $f_x(l) = \langle l, x \rangle$ for all $l \in E$. This ϕ is linear and 1–1. In fact, it is easy to verify that ϕ is an isometric imbedding into $L(E, F)$ by the proof of Lemma 3.2. Let $X' = \phi(X)$ and $K' = \phi(K)$. Then $\phi: X \to X'$ is a homeomorphism. We define $T: X' \rightrightarrows E$, $C: X' \rightrightarrows F$ and $W: X' \rightrightarrows F$ to be the same as in the proof of Theorem 3.2. The proof is organized in the following parts.

(i) $T: X' \rightrightarrows E$ is an upper semicontinuous multifunction such that $T(f)$ is a nonempty compact subset of E for all $f \in X'$, and the range $T(X')$ is contained in a compact subset of E. This is immediate from the fact that $T = T_1 \circ \phi^{-1}$ and the given hypothesis on T_1.

(ii) The graph $\text{Gr}(W)$ of W is closed in $X' \times F$, where $L(E, F)$ is endowed with the topology of bounded convergence. Indeed, let $\{f_{x_i}\}$ be a net in X' convergent to $f_x \in X'$, with respect to the topology of bounded convergence in $L(E, F)$. Clearly the norm topology and that of bounded convergence on $L(E, F)$ coincide. Let $w_i \in W(f_{x_i}) = W_1(x_i)$ such that $w_i \to w$ in F. Since ϕ is a homeomorphism, $\phi^{-1}(f_{x_i}) = x_i \to x = \phi^{-1}(f_x)$. Because the graph $\text{Gr}(W_1)$ of W_1 is closed in $X \times Z$, we have $w \in W_1(x) = W(f_x)$. This implies that $\text{Gr}(W)$ is closed in $X' \times F$.

(iii) By the hypothesis, it can be readily checked that for each finite subset N' of X', there exists a nonempty compact convex subset $L_{N'}$ of X' containing N' such that for each $f_x \in L_{N'} \setminus K'$, there exists $f_y \in L_{N'}$ satisfying

$$\langle f_y - f_x, s \rangle \in C(f_x) \quad \text{for all } s \in T(f_x).$$

It follows directly from Theorem 3.3 that there exists $f_{x_0} \in X'$ such that for each $f_x \in X'$, there is $t \in T(f_{x_0})$ with $\langle f_x - f_{x_0}, t\rangle \notin C(f_{x_0})$. Therefore, there exists $x_0 \in X$ such that

$$\forall x \in X, \quad \exists t \in T_1(x_0) \quad \text{with } \langle t, x - x_0\rangle \notin -\text{int}\, C_1(x_0).$$

This completes the proof. □

4. Generalized Vector Quasi-Variational Inequality with Operator Solutions

As an application of Lemma 2.2, we first derive an existence of solutions for (GOQVVI) as follows.

THEOREM 4.1. *Let X' be a nonempty convex subset of $L(E, F)$, $A: X' \rightrightarrows X'$ a multifunction such that each $A(f)$ is nonempty convex and each $A^{-1}(g)$ is open in X', and cl $A: X' \rightrightarrows X'$ is upper semicontinuous. Let $T: X' \rightrightarrows E$ be a multifunction such that $T(f)$ is nonempty for all $f \in X'$. Let $W: X' \rightrightarrows F$ be defined by $W(f) = F \setminus C(f)$ such that the graph $Gr(W)$ of W is closed in $X' \times F$.*
Furthermore, assume that $L(E, F)$ is endowed with a topology having the following convergence condition :
if $(f_\lambda)_{\lambda\in\Gamma} \to f$ and $x_\lambda \in T(f_\lambda)$, then there exist $x \in T(f)$ and subnets $(x_\mu), (f_\mu)$ of $(x_\lambda), (f_\lambda)$, respectively, such that $x_\mu \to x$ and $\langle f_\mu, x_\mu\rangle \to \langle f, x\rangle$.
Then (GOQVVI) is solvable.

Proof. We first define a multifunction $P: X' \rightrightarrows X'$ to be

$$P(f) := \{g \in X' \mid \forall x \in T(f),\ \langle g - f, x\rangle \in C(f)\}.$$

The proof is organized in the following parts.

(i) Clearly $P(f)$ is convex for all $f \in X'$ and P has no fixed point.
(ii) For each $g \in X'$, $P^{-1}(g)$ is open in X'. In fact, let $\{f_\lambda\}_{\lambda\in\Gamma}$ be a net in $(P^{-1}(g))^c$ convergent to $f \in X'$. Then $g \notin P(f_\lambda)$ for all $\lambda \in \Gamma$ and hence for some $x_\lambda \in T(f_\lambda)$, we have $\langle g - f_\lambda, x_\lambda\rangle \notin C(f_\lambda)$; thus $\langle g - f_\lambda, x_\lambda\rangle \in W(f_\lambda)$.
By the convergence condition, there exist $x \in T(f)$ and subnets $(x_\mu), (f_\mu)$ of $(x_\lambda), (f_\lambda)$, respectively, such that $x_\mu \to x$ and $\langle f_\mu, x_\mu\rangle \to \langle f, x\rangle$. Since $(f_\mu, \langle g - f_\mu, x_\mu\rangle) \in \text{Gr}(W)$, by virtue of the closedness of $\text{Gr}(W)$, we have $(f, \langle g - f, \bar{x}\rangle) \in \text{Gr}(W)$, that is, $\langle g - f, \bar{x}\rangle \notin C(f)$. Hence $g \notin P(f)$, so $f \in (P^{-1}(g))^c$. This shows that $(P^{-1}(g))^c$ is closed, therefore $P^{-1}(g)$ is open in X'.

Hence, all hypotheses of Lemma 2.2 are satisfied so that there must be an $f_0 \in X'$ such that $f_o \in cl\ A(f_0)$ and $A(f_0) \cap P(f_0) = \emptyset$, namely, for all

$f\in A(f_0)$, there exists an $x_0\in T(f_0)$ such that

$$\langle f-f_0,x_0\rangle\notin C(f_0).$$

Therefore, f_0 is a solution of (GOQVVI). This completes the proof. □

REMARK 4.1. Note that the convergence condition in Theorem 4.1 is a paticular case of that of Khanh and Luu [8, Theorem 2.1]. Actually the convergence condition above is a weaker one.

As an immediate consequence of Theorem 4.1, we have the following.

COROLLARY 4.1. *Let X' be a nonempty compact convex subset of $L(E,F)$, where $L(E,F)$ is endowed with the topology of bounded convergence. Let $A\colon X'\rightrightarrows X'$ be the same as in Theorem 4.1. Let $T\colon X'\rightrightarrows E$ be an upper semi-continuous multifunction such that $T(f)$ is nonempty compact for all $f\in X'$. Let $W\colon X'\rightrightarrows F$ be defined by $W(f)=F\backslash C(f)$ such that the graph $Gr(W)$ of W is closed in $X'\times F$. Then (GOQVVI) is solvable.*

In compact cases, we can derive the following existence theorem for solutions of generalized quasi-variational inequality from Corollary 4.1.

THEOREM 4.2. *Let Y and Z be normed spaces. Let X be a nonempty compact convex subset of Y and $C_1\colon X\rightrightarrows Z$ be a multifunction such that for each $x\in X$, $C_1(x)$ is a convex cone in Z with $\operatorname{int}C_1(x)\neq\emptyset$ and $C_1(x)\neq Z$. Let $A_1\colon X\rightrightarrows X$ be a multifunction such that for each $x,\ y\in X$, $A_1(x)$ is nonempty convex, ${A_1}^{-1}(y)$ is open in X, and $cl\,A_1\colon X\rightrightarrows X$ is upper semicontinuous. Let $T_1\colon X\rightrightarrows L(Y,Z)$ be an upper semicontinuous multifunction with nonempty compact values where $L(Y,Z)$ is the normed space of the continuous linear operators between Y and Z with the usual norm. Let $W_1\colon X\rightrightarrows Z$ be defined by $W_1(x)=Z\setminus -\operatorname{int}C_1(x)$ such that the graph $Gr(W_1)$ of W_1 is closed in $X\times Z$. Then there exists $x_0\in X$ such that $x_0\in cl\,A_1(x_0)$ and*

$$\forall x\in A_1(x_0),\quad \exists t\in T_1(x_0)\quad \textit{with}\ \langle t,x-x_0\rangle\notin -\operatorname{int}C_1(x_0).$$

Proof. Following the proof of Theorem 3.4, we first assume that $T\colon X'\rightrightarrows E$, $C\colon X'\rightrightarrows F$ and $W\colon X'\rightrightarrows F$ to be defined under the same circumstances as in the proof of Theorem 3.4. Moreover, the multifunction $A\colon X'\rightrightarrows X'$ is defined as follows:

$$A(f_x)=\phi(A_1(x))\ \text{for all}\quad f_x\in X'.$$

The proof is organized in the following parts.

(i) As is well-known, the topology of bounded convergence on $L(E, F)$ coincides with the usual norm topology on $L(E, F)$ in this case. So we see that $X' = \phi(X)$ is a compact convex subset of the normed space $L(E, F)$ because ϕ is a homeomorphism.

(ii) For each $f_x,\ f_y \in X', A(f_x)$ is nonempty convex, $A^{-1}(f_y)$ is open in X', and $cl\, A: X' \rightrightarrows X'$ is upper semicontinuous. Indeed, clearly $A(f_x)$ is nonempty convex because ϕ is linear. In addition, we have

$$\begin{aligned} A^{-1}(f_y) &= \{f_x \in X' \mid f_y \in A(f_x)\} \\ &= \phi(\{x \in X \mid y \in A_1(x)\}) \\ &= \phi({A_1}^{-1}(y)). \end{aligned}$$

This means that $A^{-1}(f_y)$ is open in X'. Being ϕ a homeomorphism, it should hold

$$\phi \circ (cl\, A_1) = (cl\, A) \circ \phi.$$

Hence $cl\, A = \phi \circ (cl\, A_1) \circ \phi^{-1}$. The upper semicontinuity of $cl\, A$ directly comes from that of $cl\, A_1$.

(iii) $T: X' \rightrightarrows E$ is an upper semicontinuous multifunction such that $T(f_x)$ is a nonempty compact subset of E for all $f_x \in X'$. This is immediate from the fact that $T = T_1 \circ \phi^{-1}$ and the given hypothesis on T_1.

(iv) By the step (ii) of the proof of Theorem 3.4, $W: X' \rightrightarrows F$ defined by $W(f_x) = F \setminus C(f_x)$ has a closed graph in $X' \times F$.

Thus the whole assumptions of Corollary 4.1 are satisfied so that there exists $f_{x_0} \in X'$ such that $f_{x_0} \in cl\, A(f_{x_0})$ and

$$\forall f_x \in A(f_{x_0}),\ \ \exists t \in T(f_{x_0}) \text{ with } \langle f_x - f_{x_0}, t\rangle \notin C(f_{x_0}).$$

Therefore, there exists $x_0 \in X$ such that $x_0 \in cl\, A_1(x_0)$ and

$$\forall x \in A_1(x_0),\ \ \exists t \in T_1(x_0) \text{ with } \langle t, x - x_0\rangle \notin -\mathrm{int}\, C_1(x_0).$$

This completes the proof. □

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